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If you have any questions, please do not hesitate to contact us.

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Verification of Fast Multi-pole Technique Using Finite Element Method

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Abstract
Boundary Element Method is a numerical method used to solve integral equations using direct solver to produce an accurate results, which is considered one of its advantages. Its main disadvantage that it requires large computational efforts especially for large-size problems with complex geometries. Fast Multi-pole technique was proposed to reduce the needed computational efforts. In this paper, verifications of fast multi-pole technique based on boundary element method (BEM) against finite element commercial software and analytical solution are presented. Two numerical examples were used for verification purposes; cantilever plate and variable width fixed-end plate. Results show good agreement between both methods. When comparing the needed computational memory and time, it is found that the proposed fast multi-pole technique required less memory and time. Final conclusion that fast multi-pole technique overcomes one of the most important constraints against spreading the use of Boundary Element Method.

Keywords:
Boundary Element Method, Fast Multi-pole, Finite Element Method, Iterative Solver, Taylor Expansion, Plate Bending

1. Introduction

Plate bending problem is one of the main element in structural engineering(Timoshenko & Woinowsky-Krieger, n.d.). Many numerical methods used by engineering today to solve this problem with acceptable accuracy and time(Naga & Rashed, 2016; Y. F. Rashed, Aliabadi, & Brebbia, 1997; Xiao-Yan, 1995; Zienkiewicz, Taylor, Nithiarasu, & Zhu, n.d.). The Boundary Element Method (BEM) (Brebbia & Dominguez, 1977; Long, Brebbia, & Telles, 1988)is a numerical method that is used to solve integral equation for many applications to find a numerical solution (Weeën, 1982) and PLPAK (Y. Rashed, International, & 2012, n.d.) program based on this method, similar to other methods such as Finite Element Method (FEM) which used by several commercially programs (such as SAP(SAP2000, 2011), ANSYS(ANSYS, 2005), etc.) and Finite Difference Method (FDM). Relation between BEM and other numerical solution methods is given in Figure 1 (Yijun, 2009). BEM differs than FEM and FDM that only discretization of the boundaries is needed (Liu & Nishimura, 2006). However, the BEM is similar to both methods that it uses elements, nodes, and shape functions, but on the boundaries only. BEM is used as it has several advantages over any other numerical solution method that it uses direct solver such as Gauss Elimination or Lower-Upper decomposition to find the solution of system of equations; it simplify the modeling procedure of the problem; and its produced mesh is much smaller in size. Other important advantage of the BEM method that it gives more accurate results especially for stress concentration problems as BEM is considered semi-analytical method (Yijun, 2009).

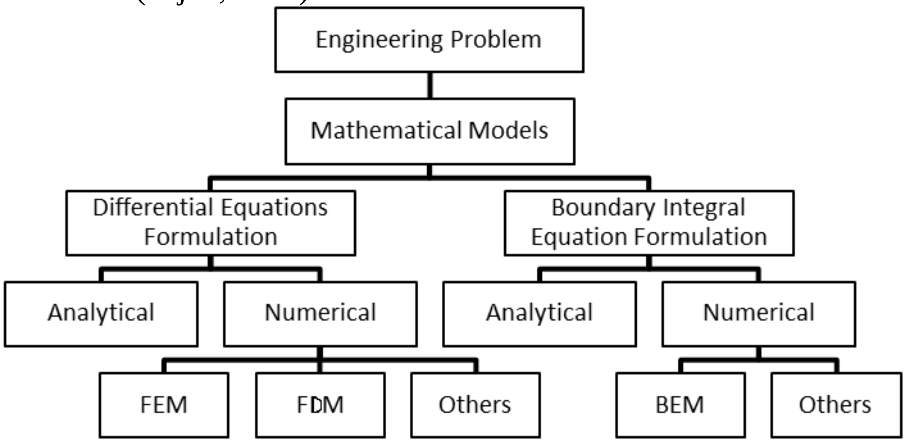

Figure**1** : Relation between BEM and other numerical solution methods (Yijun, 2009)

The main constraint of spreading the use of BEM that it creates a fully populated matrix and non-symmetrical elements that necessitate the need of larger computational efforts especially memory(Huang & Liu, 2013; Samaan, Nassar, & Rashed, 2015a; Yijun, 2009). This problem is significant especially for large size problems with complex geometries and large number of restraints. Fast Multi-Pole Method (FMM) technique (Barnes & Hut, 1986; Chen & Xiao, 2012; Greengard & Rokhlin, 1997; Popov & Power, 2001)was introduced in many fields of numerical solution to overcome the need of large computational memory. This technique when applied to BEM enables complex problems to be solved with acceptable amount of computational memory. FMM compromises of grouping the BEM mesh nodes into FMM mesh nodes, each cell in FMM mesh has a unique node at its center, to produce a banded matrix instead of the fully populated matrix produced by the BEM(Yijun, 2009). The produced banded matrix width can be controlled during the solution by controlled the number of element per leaf (the smallest cell in tree which created by FMM mesh). The main point when FMM technique applied to the BEM that an iterative solver such as GMRES should be used to find the numerical solution(Bulgakov, Białecki, & Kuhn, 1995; Hribersek & Skerget, 1996; Mansur, Araújo, & Malaghini, 1992; Prasad, Kane, Keyes, & Balakrishna, 1994; Schultz & Saad, 1986; Urekew & Rencis, 1993).

Samaan et al. (2015) applied the FMM technique to get solution of Reissner’s shear deformable plate bending problems (Karam & Telles, 1988) and proposed a second shift FMM to be applied on BEM to solve complex plate bending problems. They provided a solution for thick plate bending problem and it was found that by applying second shift FMM, a significant decrease in the computational time was attained (Samaan, Nassar, & Rashed, 2015b). The main aim of this research is to compare between FEM and BEM equipped with Second Shift FMM (FMM- BEM) solver. The FEM is commonly used in many fields to solve complex geometries and problems in a comparable computational time. The FEM results are widely accepted in many fields and by many researchers. Within this research, several structural problems were solved by both FEM and FMM-BEM. Comparison between both methods were carried out, the comparison criteria included computational time, computation memory, straining actions, and structural deformations.

2. FMM Meshing Procedures

Before applying FMM, the geometry boundary is discretized into certain number of elements same way as conventional direct BEM meshing procedures. To apply the FMM technique, a large square is created that cover the whole boundary of the problem which is called zero-level of meshing. The square is divided into smaller squares till a specific level of mesh geometry is reached which limited by number of element per leaf. Each time of division create a new level of meshing. The final resulted squares are known as cell, each cell contains specific number of originally created boundary elements. The cells that is created in the highest level of the tree is called leaf cell. Number of tree levels and number of elements in each leaf controls the width of the produced banded matrix. FMM meshing procedure is illustrated in details in (Yijun, 2009)

The produced mesh is then used to solve the problem. First, an exclusion of all cells that doesn't has elements from the tree. Definition of elements inside each cell and relation between adjacent cells are determined. Based on the relation between cells and definition of elements, definition of near field and far field for each cell is carried out. To find the effect of displacement and traction of an element (field point) on another element (source point or collocation point) in near field, conventional direct BEM equations is used to avoid any truncation error in small distances (Karam & Telles, 1988). For far field, the displacement and traction expansions that is defined by potential function is applied after applying Taylor Series (Samaan et al., 2015b). Figure 2 shows Second Shift Expansion FMM for far fields.

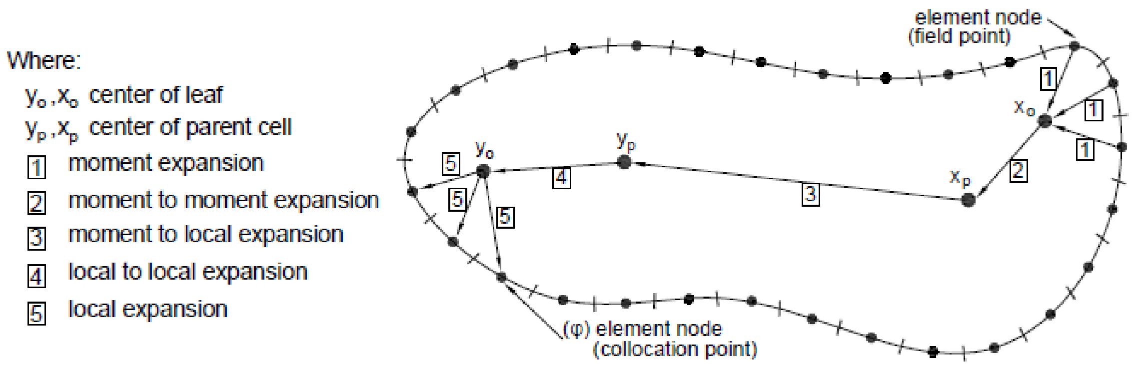

Figure2 : FMM for far field second shift expansion

3. Applying FMM technique

In the previous section, it is clear that the FMM is applied only for far field. Also, the expansion of the conventional BEM after applying Taylor Series is divided into some expansions that accumulate each other's enabling calculation of each part individually, which give the chance for parallel programing algorithm to use GPU(Tirado et al., 2014; Yokota et al., 2009). In this section, basic idea of this expansion procedure that can be applied on displacement or traction formula separately is introduced. Figure 3 shows an example application for the technique assuming a function has two independent variables (x, and y), the function define direct relation between two points; collocation and far field point, respectively. Discretization can be carried out to insert number of points between the two main points.

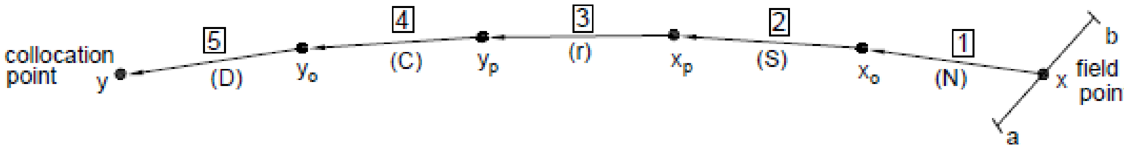

Figure3 : Example application for discretization expansion

3.1. First Shift Expansion

The first shift can be considered that the first part from expansion which leads the field point from leaf cell in the upper levels of the tree to the second lower level in the tree, this process is called the upward process. This expansion consists of two cumulative expansions; cofactor moment expansion (M), and moment to moment expansion (M2M) (Gómez & Powert, 2002). These cumulative expansions are given in the following subsections. If the problem function is represented by:

$$\varphi(y_p, x) = S_0 * \varphi(y_p, x_p) + S_i * \frac{\partial}{\partial x_i}\varphi(y_p, x_p) + S_{ij} * \frac{\partial^2}{\partial x_i \partial x_j}\varphi(y_p, x_p) + \cdots \dots \dots . \dots \dots \dots + S_{k_1 k_2 \dots k_n} * \frac{\partial^n}{\partial x_{k_1}\partial x_{k_2}\dots\partial x_{k_n}}\varphi(y_p, x_p) \quad \text{Equation 1}$$

Where, using $(r = x_0 - x_p)$ the constant which represent the co-factor of moment to moment expansion (M2M) can be expressed as:

$$S_i = C_i - r_i * C_o$$

$$S_{ij} = C_{ij} - r_j * C_i + \frac{r_i * r_j}{2!} * C_o$$

And in general the co-factor will be:

$$S_{k_1 k_2 \dots k_n} = C_{k_1 k_2 \dots k_n} - r_{k_n} * C_{k_1 k_2 \dots k_{n-1}} + \cdots + (-1)^n \frac{r_{k_1} * \dots * r_{k_n}}{n!} * C_o \quad \text{Equation 2}$$

3.2. The second shift expansion (SSE)

This shift is the second part of expansion that lead the first shift value from top level of tree to the collocation point in leaf. This expansion called downward process. This process consists of three expansions; moment to local expansion (M2L), local to local expansion (L2L), and local expansion (L). The function after applying the second shift can be represented as follows:

- The Local expansion can be written as following:

$$\varphi(y, x) = C_0 + (y_0 - y)_i * C_i + \frac{(y_0 - y)_i * (y_0 - y)_j}{2!} * C_{ij} + \frac{(y_0 - y)_i * (y_0 - y)_j * (y_0 - y)_k}{3!} * C_{ijk} + \cdots + \frac{(y_0 - y)_{k_1} * (y_0 - y)_{k_2} * \dots * (y_0 - y)_{k_n}}{n!} * C_{k_1 \dots k_n} \quad \text{Equation 3}$$

Where are the coefficients of equation (3) which represent the local expansion can be obtained from local to local expansions which written as following:

- The Local to Local expansion:

$$C_0 = \varphi(y_p, x) + (y_p - y_0)_i * \frac{\partial}{\partial x_i}\varphi(y_p, x) + \frac{(y_p - y_0)_i * (y_p - y_0)_j}{2!} * \frac{\partial^2}{\partial x_i \partial x_j}\varphi(y_p, x) + \frac{(y_p - y_0)_i * (y_p - y_0)_j * (y_p - y_0)_k}{3!} * \frac{\partial^3}{\partial x_i \partial x_j \partial x_k}\varphi(y_p, x) + \cdots$$

$$C_i = \frac{\partial}{\partial x_i}\varphi(y_p, x) + (y_p - y_0)_i * \frac{\partial^2}{\partial x_i \partial x_j}\varphi(y_p, x) + \frac{(y_p-y_0)_i*(y_p-y_0)_j}{2!} * \frac{\partial^3}{\partial x_i \partial x_j \partial x_k}\varphi(y_p, x) + \cdots$$

$$C_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}\varphi(y_p, x) + (y_p - y_0)_i * \frac{\partial^3}{\partial x_i \partial x_j \partial x_k}\varphi(y_p, x) + \cdots \qquad \text{Equation 4}$$

And can proceed in this sequence if we need to get any other coefficients.

Also, the function which appeared in local to local expansion leads us to represent the moment to local expansion which represent as following:

- The Moment to Local expansion:

$$\varphi_{,i}(y_p, x) = S_0 * \frac{\partial}{\partial x_i}\varphi(y_p, x_p) + S_i * \frac{\partial^2}{\partial x_i \partial x_j}\varphi(y_p, x_p) + S_{ij} * \frac{\partial^3}{\partial x_i \partial x_j \partial x_k}\varphi(y_p, x_p) + \cdots$$

$$\varphi_{,ij}(y_p, x) = S_0 * \frac{\partial^2}{\partial x_i \partial x_j}\varphi(y_p, x_p) + S_i * \frac{\partial^3}{\partial x_i \partial x_j \partial x_k}\varphi(y_p, x_p) + \cdots \qquad \text{Equation 5}$$

And can proceed in this sequence if we need to get any other coefficients.

The above expansions with its first and second shift can be applied in both displacement and traction as a function in moment and shear as it represented in details in (Samaan et al., 2015b). The above procedure is summarized in Figure 4.

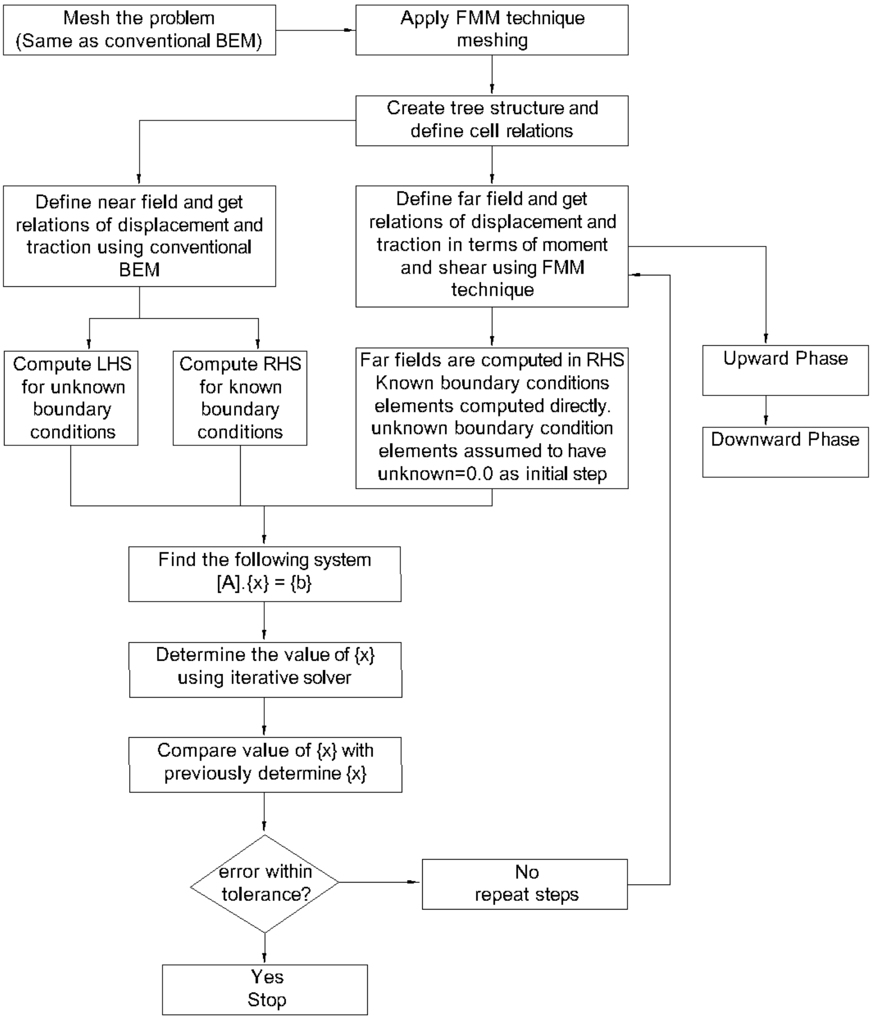

Figure4 : FMM technique summary flow chart

4. Verification Examples

Two numerical examples were used to verify the proposed FMM technique based on BEM against FEM. The two examples were selected to provide different geometry, boundary conditions, and loading conditions. Three terms of Taylor Series were used in the verification, as by increasing number of terms; accuracy of the results increase but more computational time is needed. These three terms are considered enough for the verification purposes. Element size and mesh geometry has influence on both FEM and FMM-BEM results, element size in both methods are kept constant.

4.1. Cantilever Plate

The cantilever rectangular plate as shown in Figure 5, is considered in this example. The plate is subjected to a uniform distributed load with intensity -1.5 KN/m'. The young's modulus for the plate material is E=3 x 10^7 Mpa and poisson's ratio ν is set to zero to enable comparison with simple analytical solution of the beam theory. Figure 6 demonstrate the deflection along the cantilever plate center line. Results are plotted from the proposed new FMM-BEM together with the traditional finite element method (SAP2000, 2011). Also as shown in Tables 1 and 2, the results of deflection and rotations were recorded at point (A), also shear and moment were recorded at point (B), respectively are compare with the analytical solution. Five meshes are considered in each method as mentioned in tables. It can be concluded that acceptable accuracy were attained by using FMM-BEM with comparable computational time which represent in Figure 7.

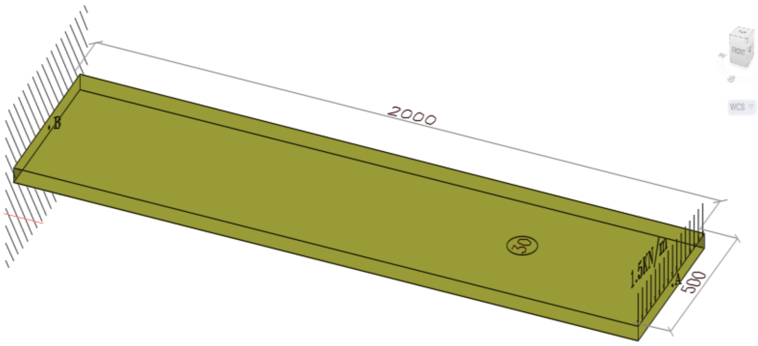

Figure5 : The rectangular cantilever plate considered in example (1)

Table 1: Deflection and rotation results at Point (A)

Element Length (m)	No. of elements		Rotation (Rad.)			Deflection (mm)		
	FMMBEM	Finite Ele. (SAP)	Analytical	FMMBEM	Finite Ele. (SAP)	Analytical	FMMBEM	Finite Ele. (SAP)
1.00	50	100		0.004389	0.004440		-59.308	-59.260
0.50	100	400		0.004555	0.004469		-59.868	-59.751
0.25	200	1600	0.004444	0.004569	0.004453	-59.259	-59.644	-59.429
0.125	400	6400		0.004424	0.004447		-59.171	-59.326
0.0625	800	25600		0.004411	0.004446		-59.013	-59.291

Table 2: Shear and Moment results at Point (B)

Element Length (m)	No. of elements		Shear (KN/m')		Moment (KN.m/m')	
	FMMB EM	Finite ele. (SAP)	FMMB EM	Finite ele. (SAP)	FMMB EM	Finite ele. (SAP)
1.00	50	100	1.998	1.500	-30.466	-30.001
0.50	100	400	1.802	1.341	-30.401	-31.109
0.25	200	1600	1.638	1.650	-30.172	-30.364
0.125	400	6400	1.615	1.516	-30.107	-30.086
0.0625	800	25600	1.616	1.504	-30.0899	-30.0192

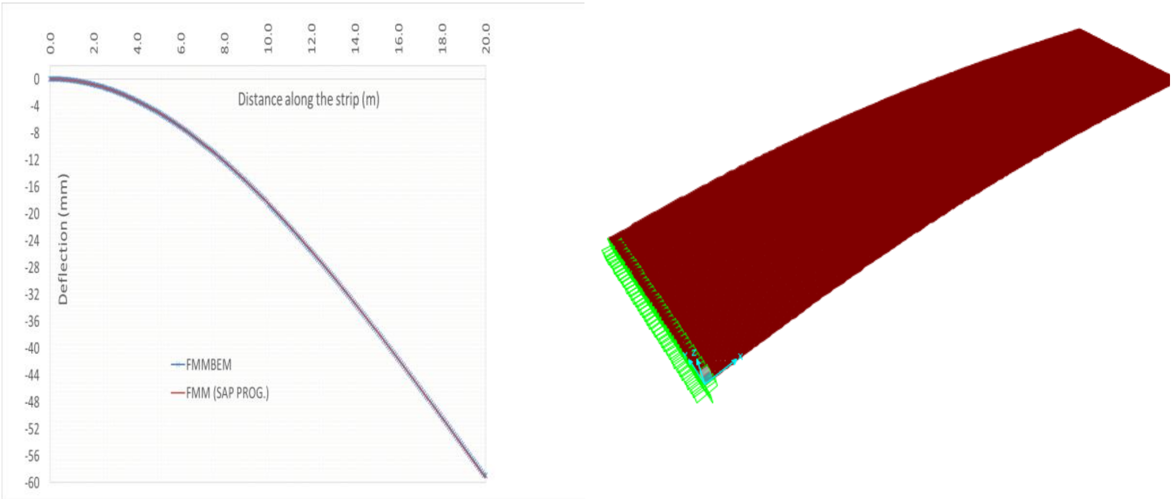

Figure6 : Vertical displacements of plate in example (1) due to bending load

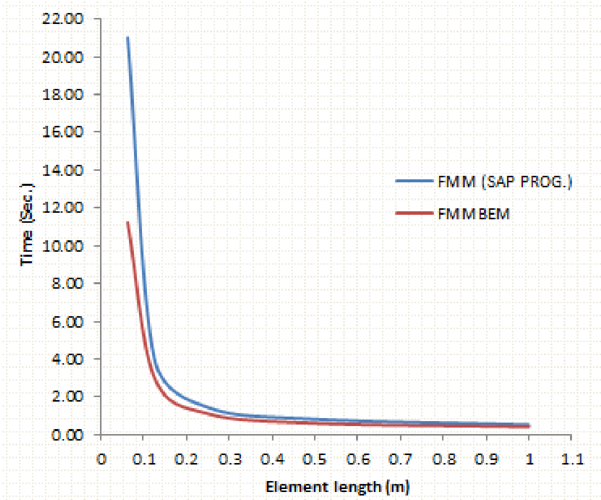

Figure7 : Time comparison according to different meshes depends on element length

4.2. Variable Width Rigidly Supported Plate

Rectangular plate of 15.0m length and variable width that is rigidly supported along both ends is used as shown in Figure 8. The plate is subjected to uniformly distributed load of 10 kN/m' in both edge of mid span. Material properties that are used in this verification example are: Young's Modulus = $3x10^7$ Mpa; and Poisson's ratio = 0.1.The displacement and rotations were recorded at Point A, Bending Moment and Shear is recorded at Point B. The problem was modeled in both SAP2000 software and using the proposed FMM-BEM and summary of results as shown in Tables 3 and 4, the results of deflection and rotations were recorded at point (A), also shear and moment were recorded at point (B) as shown in Figure 9. It can be concluded that acceptable accuracy were attained by using FMM-BEM with comparable computational time and memory usage as shown in Figure 10. Further accuracy of results can be attained in FMM-BEM by increasing number of terms in Taylor Series.

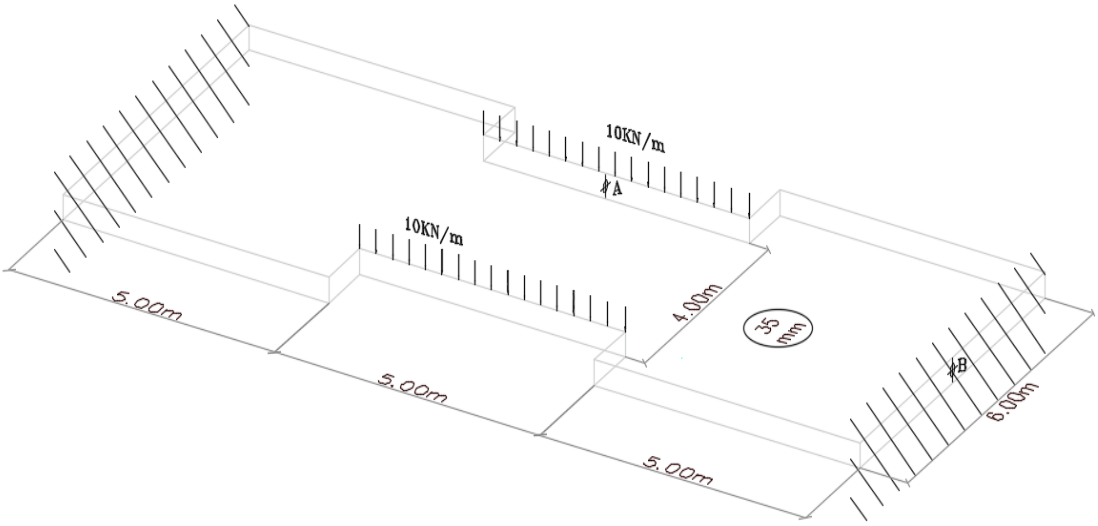

Figure**8** : The rectangular rigidly supported plate considered in example (2)

Table 3: Deflection and rotation results at Point A

Element Length (m)	No. of elements		Rotation (Rad.)		Deflection (mm)	
	FMMB EM	Finite Ele. (SAP)	FMMB EM	Finite Ele. (SAP)	FMMB EM	Finite Ele. (SAP)
1.00	46	80	-0.000113	-0.000116	3.226	4.747
0.50	92	320	-0.000111	-0.000119	3.104	3.156
0.25	184	1280	-0.000109	-0.000118	3.104	3.105
0.125	368	5120	-0.000111	-0.000118	3.093	3.090
0.0625	736	20480	-0.000115	-0.000118	3.088	3.085

Table 4: Shear and Moment results at Point B

Element Length (m)	No. of elements		Shear (KN/m')		Moment (KN.m/m')	
	FMMB EM	Finite Ele. (SAP)	FMMB EM	Finite Ele. (SAP)	FMMB EM	Finite Ele. (SAP)
1.00	46.00	80	10.144	8.350	33.514	32.687
0.50	92.00	320	9.508	9.894	33.712	34.746
0.25	184.00	1280	9.135	9.090	33.740	34.015
0.1250	368.00	5120	9.027	8.932	33.741	33.796
0.0625	736.00	20480	8.993	8.840	33.737	33.745

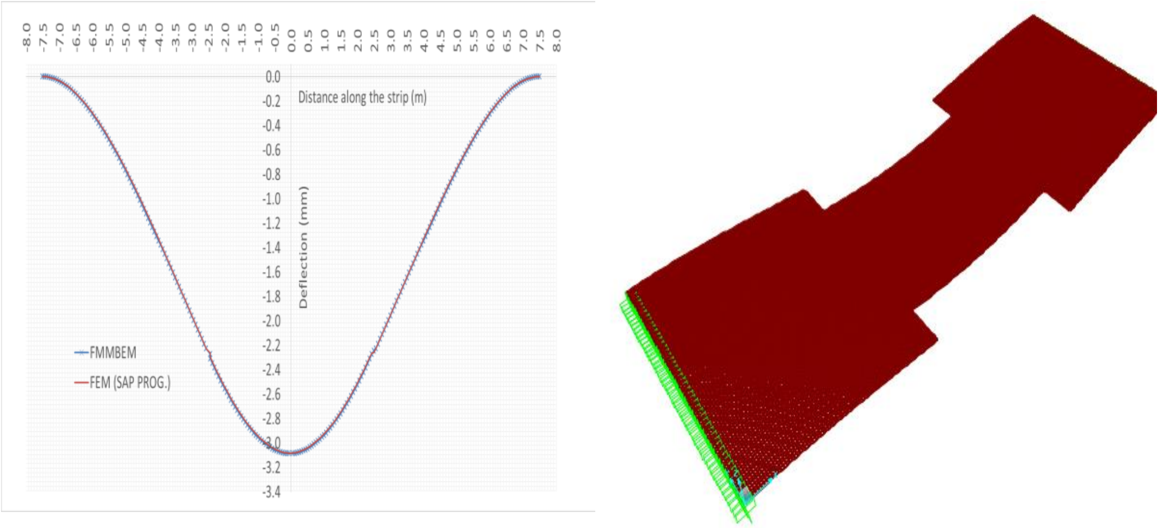

Figure**9** : Vertical displacements of plate in example (2) due to bending load.

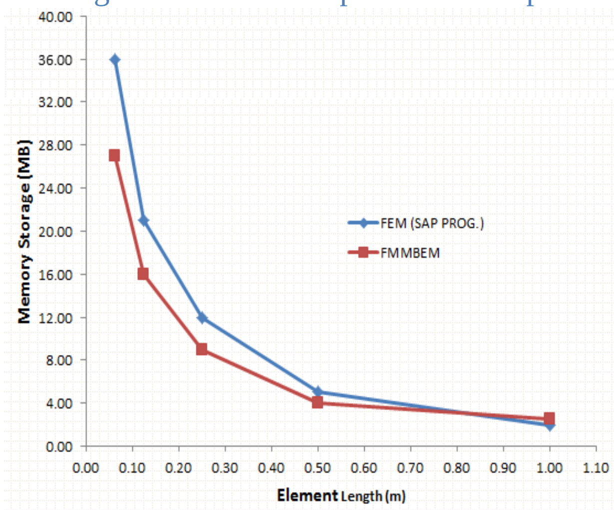

Figure**10 :** Memory storage comparison according to different meshes depends on element length

5. Conclusions

In this paper, the fast multi-pole technique after applied to BEM become capable to competing the finite element method for solving large scale problems with suitable usage of memory and acceptable accuracy of results as proved in the verification examples discussed in this paper. These results obtained with three terms of Taylor's series only to achieve consistency between time of solution, memory usage and results' accuracy. When comparing the needed computational memory and time, it is found that the proposed fast multi-pole technique required less memory and time. It can be concluded that fast multi-pole technique overcome one of the most important constraint against spreading the use of Boundary Element Method by reducing the needed computation efforts. Also, Fast Multi-pole technique provided the direct boundary element solver with the ability to use parallel processing and GPU computing techniques.

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